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A POINTWISE ERGODIC THEOREM FOR THE GROUP OF RATIONAL ROTATIONS[1,2]

BY

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ABSTRACT. Let f be a bounded, measurable function defined on the multiplicative group Ω of complex numbers of absolute value 1, and define

$$f_n(\omega) = \frac{1}{n} \sum_{i=1}^{n} f(z_n^i \omega), \qquad \omega \in \Omega, \tag{1}$$

where z_n is a primitive nth root of unity. The present paper generalizes this result of Jessen [1934]: if $n(k)$ is an increasing sequence of positive integers with $n(k)$ dividing $n(k')$ whenever $k < k'$, then $f_{n(k)}$ converges almost surely as $k \to \infty$.

1. Introduction. For f a real-valued function defined on the unit circle Ω, let

$$f_n(\omega) = \frac{1}{n} \sum_{i=1}^{n} f(z_n^i \omega), \tag{1.1}$$

where $\omega \in \Omega$ and z_n^i is the ith power of z_n, a primitive nth root of unity.

Jessen [1934] raised, but did not answer, the question whether, for integrable f, f_n converges almost surely. He did show that f_n converges almost surely provided n ranges only over the elements of a *chain*, that is, an infinite subset K of the positive integers, N, with the property that each element of K is a divisor of the next. Jessen's question was not answered until Rudin [1964] exhibited a bounded, measurable function f for which f_n diverges everywhere.

For each set $\mathcal{Q}$ of functions f, a subset K of N is an *$\mathcal{Q}$-set* if, for every $f \in \mathcal{Q}$, for k restricted to K, f_k converges a.s. as $k \to \infty$. For $k \in N$, $l \in N$, let $k \vee l$ be the least common multiple of k and l and, for $K \subset N$, $L \subset N$, let $K \vee L$ be the set of all $k \vee l$ for $k \in K$, $l \in L$. Let $\mathfrak{M}$ be the set of bounded, measurable f. Baker [1976] discovered that if K and L are $\mathfrak{M}$-sets, then so is $K \vee L$. The *dimension* of a nonempty set K of positive integers is the least positive integer d such that there are chains $K_1, \ldots, K_d$ for which K is a subset of $K_1 \vee \cdots \vee K_d$. As is an immediate consequence of Jessen's

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and Baker's results together, every set K of finite dimension is an $\mathfrak{M}$-set. More is true, namely, every such K is an $\mathcal{L}_p$-set for every $p > 1$, which is a principal purpose of the present paper to demonstrate. Somewhat more sharply:

THEOREM 1.1. *For each positive integer d, if K has dimension d, then K is an $\mathcal{L} \log^{d-1} \mathcal{L}$ set.*

As usual, $\mathcal{L} \log^{d-1} \mathcal{L}$ is the set of all f such that $\phi_d \circ f$ is integrable, where $\phi_d: R \to R^+$ is defined by $\phi_d(x) = |x|(\log|x|)^{d-1}$ if $|x|$ exceeds 1 and is zero otherwise.

That $\mathcal{L} \log^{d-1} \mathcal{L}$ in Theorem 1.1 cannot be replaced by $\mathcal{L}$ can be seen by consideration of the set K of dimension 2 which consists of the integers of the form $2^i 3^j$ (Baker [1976]). We presume that for no $d \geq 2$ can $\mathcal{L} \log^{d-1} \mathcal{L}$ be replaced by $\mathcal{L} \log^{d-2} \mathcal{L}$.

A set K of positive integers has *breadth* at most d if the least common multiple of every finite subset of K is the least common multiple of at most d elements of that subset; the least such d is the *breadth* of K and, if no such d exists, K has *infinite breadth*. Let $\mathfrak{B}$ and $\mathfrak{D}$ be the collection of K of finite breadth and of finite dimension, respectively, and let $\mathcal{C}$ be the collection of K which are $\mathfrak{M}$-sets ('$\mathcal{C}$' for 'convergence'). Then $\mathfrak{B} \supseteq \mathcal{C} \supseteq \mathfrak{D}$, where the first inclusion was established by Rudin [1964], and the second, as noted above, follows from Jessen's and Baker's results together. An example provided in §3 shows that the inclusion $\mathfrak{B} \supset \mathfrak{D}$ is strict, but we leave open the interesting question whether $\mathcal{C}$ is identical with $\mathfrak{B}$, or with $\mathfrak{D}$, or with neither.

A generalization of Theorem 1.1 to groups which are not necessarily commutative is offered in §4, and an application is then made to a law of large numbers for multiparameter, semiexchangeable processes.

2. Proof of Theorem 1.1. For $f \in \mathcal{L}$, let f_n be as in (1.1) and define E_n by $E_n f = f_n$. Give Borel subsets of Ω the usual uniform probability measure. Then E_n acts as a conditional expectation operation given $\mathfrak{F}_n$, the σ-field of Borel sets F such that $\omega \in F$ implies $z_n^i \omega \in F$ for all $1 \leq i \leq n$. As is easily verified,

$$E_m E_n = E_n E_m = E_{m \vee n}, \qquad m, n \in N,$$

which is an expression of the fact that the σ-fields $\mathfrak{F}_m$ and $\mathfrak{F}_n$ are conditionally independent given $\mathfrak{F}_m \cap \mathfrak{F}_n$ (see II.T.45 of Dellacherie and Meyer [1975]).

Suppose that K equals $K_1 \vee \cdots \vee K_d$ where each K_r is a chain. Let $K_r(n)$ be the nth element of K_r and, for $i \in N^d$, let $k(i)$ be the least common multiple of the d integers $K_r(i(r))$, $1 \leq r \leq d$. For $i \in N^d$, set E_i equal to $E_{k(i)}$. Verify that the E_i, $i \in N^d$, are conditional expectation operators which satisfy

$$E_iE_j = E_{i\vee j}, \qquad i \in N^d, j \in N^d, \tag{2.1}$$

where $i \vee j \in N^d$ is the supremum of i and j in the coordinatewise ordering of N^d.

Condition (2.1) already implies that for $f \in \mathfrak{L} \log^{d-1} \mathfrak{L}$, E_if converges almost surely to a limit $E_\infty(f)$. This means that, setting a null set aside, for each $\varepsilon > 0$, there is an integer l such that for all $i \in N^d$ each of whose coordinates exceeds l, $|E_if - E_\infty f| < \varepsilon$. But the E_if converge in a stronger sense.

A countable array e_i of real numbers converges *rearrangeably* to a real number e_∞ if, for some (and hence every) sequential ordering of the e_i, the sequence converges to e_∞.

Let $i(r, n)$ be that element of N^d which has all of its coordinates equal to 1, except the rth, which equals n. For each r, the set of all $i(r, n)$ as n ranges over the positive integers is the rth *axis*. Let $E_{r,\infty}$ be the infimum of E_i as i ranges over the rth axis. Here, the *infimum* of a collection of conditional expectation operators is the conditional expectation operator given the intersection of the corresponding σ-fields.

As is well known and easily verified, an $\mathfrak{L}_1$-function which is invariant under a dense set of rotations is constant almost certainly. (A formal argument may be seen in [Dubins, 1977, Lemma 2].) Therefore,

$$\begin{gathered}\text{for each } r,\ 1 \leq r \leq d, \text{ and each } f \in \mathfrak{L}_1,\\ E_{r,\infty} f \text{ is a constant almost surely.}\end{gathered} \tag{2.2}$$

As will soon be seen, (2.1) and (2.2) together imply that for each $f \in \mathfrak{L} \log^{d-1} \mathfrak{L}$, there is a null set outside of which E_if converges rearrangeably to the constant Ef.

Verify that as i ranges over N^d, $k(i)$ ranges over all of K, so the range of the array $(E_if,\ i \in N^d)$ is identical with the range of $(f_k,\ k \in K)$. Since the E_if converge rearrangeably almost surely, f_k converges (in the usual sense) almost surely as k ranges over K. Since there plainly was no real loss in generality in assuming that K was equal to $K_1 \vee \cdots \vee K_d$ rather than a subset of it, Theorem 1.1 is fully proved modulo only the proof of this assertion:

THEOREM 2.1. *Let $(E_i,\ i \in N^d)$ be an array of conditional expectation operators which satisfy* (2.1) *and* (2.2). *Then for any $f \in \mathfrak{L} \log^{d-1} \mathfrak{L}$, there is a null set outside of which E_if converges rearrangeably to the constant Ef.*

For Theorem 2.1, the underlying countably additive probability space can be arbitrary. An instance of Theorem 2.1 was established by Smythe [1973] following Cairoli [1970]. Priority goes also to Cairoli and Walsh [1975], Smythe [1974] and Gut [1976]. Indeed, as shown by Gut [1976] an inequality of Cairoli can be used to show that for $f \in \mathcal{L}\log^{d-1}\mathcal{L}$, condition (2.1) is sufficient for the almost sure convergence of $E_i f$, in the coordinatewise ordering. Moreover, by an argument of Smythe [1973], Theorem 2.1 can be derived from Gut's convergence theorem. It is simpler, however, to derive both theorems directly from the maximal inequality of Cairoli [1970] which, slightly reformulated, states:

There are constants a and b which depend on d only such that, if $(E_i, i \in N^d)$ *satisfies* (2.1) *then, for* $\varepsilon, \delta > 0$,

$$P\Big(\sup_i |E_i f| \geqslant \delta\Big) \leqslant \varepsilon(1 + aE\phi_d(cf)) \tag{2.3}$$

where $c = b/\varepsilon\delta$.

Let E_∞ be the infimum of the conditional expectation operators $(E_i, i \in N^d)$.

LEMMA 2.1. *Suppose* $(E_i, i \in N)$ *satisfies* (2.1), $f \in \mathcal{L}\log^{d-1}\mathcal{L}$, $E_\infty f = 0$, j: $N \to N^d$ *is increasing and the infimum of* $E_{j(n)}$ $(n \in N)$ *is* E_∞. *Then* $\sup_i |f_{j(n)+i}| \to 0$ *almost surely as* $n \to \infty$.

PROOF. Because $\sup_i |f_{j(n)+i}|$, say f_n^*, is decreasing in n, it suffices to show that $P(f_n^* \geqslant \delta) \to 0$ as $n \to \infty$. Apply (2.3) with $(f_{j(n)}, E_{j(n)+i})$ substituted for (f, E_i) to see that it is enough to show: for each $c > 0$, $E\phi(cf_{j(n)}) \to 0$ as $n \to \infty$. Fix c, put $g_n = cf_{j(n)}$ and check that $(g_n, E_{j(n)}; n \in N)$ is a one-dimensional reversed martingale which converges almost surely to $cf_\infty = 0$ as $n \to \infty$. Since $\phi_d(x) = 0$ for $|x| \leqslant 1$, $\phi_d \circ g_n \to 0$ almost surely, too. Since ϕ_d is convex, $\phi_d \circ g_n$ is a nonnegative, reversed submartingale. Therefore, by [**13**, Corollary V-3-13],

$$\lim_n E\phi_d \circ g_n = E \lim_n \phi_d \circ g_n = 0,$$

which completes the proof.

To obtain Theorem 2.1, first reduce to the case $Ef = 0$ and apply Lemma 2.1 with $j(n) = j_r(n)$ for each $r \in D$. To obtain Gut's theorem, reduce to the case $E_\infty f = 0$ and apply Lemma 2.1, with $j(n) = (n, \ldots, n)$.

3a. A divergent, two-parameter, bounded, reversed martingale. Let p_n be the nth prime, Π_n the product of the first n primes, and let Π_0 be 1. Let $i = (i_1, i_2)$ range over N^2 and define $k(i)$ thus.

$$k(i) = \begin{cases} \Pi_{m(m+1)} & \text{if } i_1 + i_2 = 2m, \\ \Pi_{m(m+1)} p_{m(m+1)+i_1+1} & \text{if } i_1 + i_2 = 2m + 1. \end{cases}$$

Since $k(i)$ divides $k(j)$ whenever i is less than j in the coordinatewise ordering, $(f_{k(i)}, \mathcal{F}_{k(i)}; i \in N^2)$ is a reversed martingale for any $f \in \mathcal{L}$, where f_n is defined by (1.1). (For $E_i = E_{k(i)}$, (2.1) fails, though $E_iE_j = E_jE_i$ and (2.2) holds.) As is easily verified, the range K of $(k(i), i \in N^2)$ has infinite breadth, so by the construction of Rudin [1964] there exists a bounded Borel function f such that f_k diverges everywhere as $k \to \infty$ through K. The array $(f_{k(i)}, i \in N^2)$ is a.s. divergent, for it fails to converge rearrangeably anywhere, though each row sequence $f_{k(i_1,\cdot)}$ converges to Ef a.s. by Jessen's theorem, as does each column sequence.

This example supplements the example of a divergent, directed martingale given by Dieudonné [1950]. Another, and much simpler example than either of these, will be offered in a forthcoming note.

3b. A set of integers of finite breadth and of infinite dimension. Let n be a positive integer, let p_n be the nth prime, let K_n be the set of all integers k such that for some $m \leqslant n$, k is the product of all, except at most one, of the first m primes, $2 = p_1 < p_2 < \cdots < p_m$, and let $K = \bigcup_n K_n$. Then K is the desired example. That K has breadth 2 is obvious. What must be demonstrated, therefore, is that it has infinite dimension. For this, it plainly suffices to show that the dimension of K_n converges to ∞ as $n \to \infty$.

A *d-scheme* is a d-tuple $\sigma = (\sigma_1, \ldots, \sigma_d)$ where each σ_r is a prime-valued function defined on a finite set D_r of integers. For each $j \in D_r$ let $\hat{\sigma}_r(j)$ be the product over all $i \in D_r$ which do not exceed j of $\sigma_r(i)$, and let $R(\sigma)$, the *reach* of σ, be the smallest set of integers which satisfies these two conditions: (a) for each r, $1 \leqslant r \leqslant d$, and each $j \in D_r$, $\hat{\sigma}_r(j) \in R(\sigma)$; (b) the least common multiple of each finite subset of $R(\sigma)$ is an element of $R(\sigma)$.

To show that the dimension of a finite set of positive integers exceeds d, it plainly suffices to show that the set is not included in $R(\sigma)$ for any d-scheme σ.

For a d-scheme $\sigma' = (\sigma_1', \ldots, \sigma_d')$, define the relation $\sigma' \leqslant \sigma$ if, for each r, $1 \leqslant r \leqslant d$, σ_r' is the restriction of σ_r to a subset D_r' of D_r.

Call σ *minimal* for K_n if these two conditions hold: (a) $K_n \subset R(\sigma)$; and (b) if $\sigma' \leqslant \sigma$ and $K_n \subset R(\sigma')$, then $\sigma' = \sigma$. Call σ *orderly* if, for each r, $1 \leqslant r \leqslant d$, and each i and j in D_r with $i < j$, $\sigma_r(i) < \sigma_r(j)$.

LEMMA 3.1. *Any scheme which is minimal for K_n is orderly.*

PROOF. Let σ be a d-scheme which is minimal for K_n. Trivially, for all r, $1 \leqslant r \leqslant d$, and all $j \in D_r$, $\sigma_r(j) \leqslant p_n$. Moreover, as is easily verified, the inequality is strict unless j is the largest element of D_r. Let σ_r' be the restriction of σ_r to the set of $j \in D_r$ for which $\sigma_r(j) < p_n$. Plainly, $K_{n-1} \subset R(\sigma')$. In fact, σ' is minimal for K_{n-1}. For otherwise, as will soon be shown, there exists a d-scheme $\rho \leqslant \sigma$, $\rho \neq \sigma$ with these two properties: (a) $K_{n-1} \subset R(\rho)$, and (b) for each $k \in K_n$, $k \notin K_{n-1}$, $\exists k^* \in R(\rho)$ such that p_n divides k^*, and k^* divides k. For such a k, $k' = k/p_n$ is plainly in K_{n-1} and hence $k' \in R(\rho)$. So k, the l.c.m. of two elements of $R(\rho)$, is itself an element of $R(\rho)$. That is, $K_n \subset R(\rho)$, which contradicts the assumption that σ is minimal for K_n.

Return to the proof that the asserted ρ does exist if σ' were not minimal for K_{n-1}. For let ρ' be a d-scheme which satisfies $\rho' \leqslant \sigma'$, $\rho' \neq \sigma'$ and $K_{n-1} \subset R(\rho')$, let ρ'' be defined by letting ρ_r'' be the restriction of σ_r to the set of $j \in D_r$ for which $\sigma_r(j) = p_n$, and let $\rho_r(j)$ be $\rho_r'(j)$ or $\rho_r''(j)$ according as j is in the domain of ρ_r' or of ρ_r''. That ρ has the desired properties is verified without difficulty, so the proof that σ' is minimal for K_{n-1} is complete. So not only is $\sigma_r(j) < p_n$ unless j is the largest element of D_r, but also $\sigma_r'(j) < p_{n-1}$ unless j is the largest element in the domain of σ_r'. A simple induction now completes the proof that σ is orderly. $\square$

Associate to each d-scheme σ and prime p the set S_p of all r such that for some j, $\sigma_r(j) = p$.

LEMMA 3.2. *Suppose that σ is an orderly scheme, p and q are primes, $p < q$, $S_p \supseteq S_q$, $k \in R(\sigma)$ and q divides k. Then p divides k.*

PROOF. Easy and omitted.

LEMMA 3.3. *Let σ be an orderly d-scheme for which $K_n \subset R(\sigma)$, and suppose that p and q are primes with $2 \leqslant p < q \leqslant p_n$. Then S_p does not include S_q. In particular, S_p and S_q are distinct, so $n \leqslant 2^d - 1$.*

PROOF. Let Π be the product of the first n primes and let k be Π divided by p. Then $k \in K_n$, so $k \in R(\sigma)$. Plainly, q divides k and p does not. Lemma 3.2 now implies the conclusion that S_p does not include S_q. $\square$

Of course, Lemmas 3.1 and 3.3 imply that the dimension of K_n is no less than $\log_2(n + 1)$, so K does not possess a finite dimension.

4. Permutable products and conditional independence. Let G be a topological Hausdorff group which is faithfully represented as a group of measure preserving transformations of a countably additive probability space $(\Omega, \mathfrak{F}, P)$ for which the mapping $(g, \omega) \to g(\omega)$ is jointly measurable. For each compact subgroup H of G, let E_H be the operator

$$(E_H X)(\omega) = \int_H X(h(\omega))\, dh, \tag{4.1}$$

where dh is normalized Haar measure on H. Of course, E_H is well defined, at least on the space of all bounded, $\mathfrak{F}$-measurable, random numbers X, and is a version of the conditional expectation operator associated with the σ-field of H-invariant, $\mathfrak{F}$-measurable, subsets of Ω.

The *product* ST of subgroups S and T of G, is the set of all products st for $s \in S$ and $t \in T$, and $S \vee T$ is the smallest subgroup of G which includes both S and T. Of course, $ST \subset S \vee T$, and the reverse conclusion holds if, and only if, $ST = TS$, in which event S and T are *permutable* and ST is the *permutable product* of S and T, a terminology borrowed from finite group theorists.

PROPOSITION 4.1. *Let S and T be compact subgroups of G. If S and T are permutable, then E_S and E_T commute. If $G = \Omega$ is compact, G acts on Ω by left translation, and P is normalized Haar measure on G, then the converse also holds.*

PROOF. As is easily verified, it suffices to give the proof in the special case–henceforth assumed– that G is a compact group acting on itself by left translation. For this special case, (4.1) becomes:

$$E_H u(g) = \int_H u(hg)\, dh, \tag{4.1*}$$

where u ranges over the bounded, Borel functions defined on G, g ranges over G and h over H.

Call u *T-invariant* if $u(tg)$ is $u(g)$ for all $t \in T$ and all $g \in G$.

LEMMA 4.1. *If $ST = TS$ and u is T-invariant and continuous, then*

$$\int_H u(h)\, dh = \int_S u(s)\, ds, \tag{4.2}$$

where dh and ds are normalized Haar measures on $H = ST$ and on S.

PROOF OF (4.2). As von Neumann's theory of invariant integration ([**12**], [**14**]) makes evident when G is second countable–and as is valid without this hypothesis–for each $\varepsilon > 0$ there exists $s_i \in S$, $i = 1, \ldots, n$, such that $(1/n)\Sigma_i\, u(ss_i)$ differs from the right-hand side of (4.2), say c, by at most ε, uniformly in $s \in S$. Since u is T-invariant, $(1/n)\Sigma_i\, u(hs_i)$, therefore, differs from c by at most ε, uniformly in $h \in TS = H$. But the only number c which has this property is the left-hand side of (4.2), so the lemma is proved.

CONTINUATION OF PROOF OF PROPOSITION 4.1. Suppose that S and T are compact and permutable. The program is to show that

$$E_{ST} f = E_S E_T f \tag{4.3}$$

for all bounded, Borel f, which plainly implies that E_S and E_T commute. Because the collection of f which satisfy (4.3) is closed under uniformly bounded, pointwise convergence, it suffices to establish (4.3) for continuous f. Since $T \subset ST$, the left-hand side of (4.3) is $E_{ST} E_T f$, so it even suffices to show:

$$E_{ST} E_T f = E_S E_T f \tag{4.4}$$

for continuous f. Moreover, since $E_T f$ is plainly T-invariant, it is enough to establish

$$E_{ST} v = E_S v \tag{4.5}$$

for all T-invariant, continuous v. For $H = ST$, $h \in H$, $g \in G$, write $v_g(h) = v(hg)$ and calculate, thus.

$$E_{ST}v(g) = \int_H v(hg)dh = \int_H v_g(h)dh$$

$$= \int_S v_g(s)ds = \int_S v(sg)ds = E_S v(g), \tag{4.6}$$

where the first equality is by definition of E_H, the second by definition of v_g, the third by Lemma 4.1 applied to v_g, the fourth by definition of v_g, and the last by definition of E_S. This proves (4.5), hence (4.3) and the first assertion of Proposition 4.1.

Consider now the converse. We thank Michael Cowling for suggesting the following demonstration, which is much simpler than our original one. For a bounded, continuous function f on G and a probability measure μ, define a function $f^*\mu$ by

$$f^*\mu(x) = \int_G f(xg)\,\mu(dg).$$

Then $E_S f = f^*\mu_S$ where μ_S is Haar measure on S. For the convolution $\mu^*\nu$ of two probability measures μ and ν defined in the usual way, $f^*(\mu^*\nu) = (f^*\mu)^*\nu$. So $E_S E_T = E_T E_S$ implies that $\mu_S^*\mu_T = \mu_T^*\mu_S$. Since S and T are the respective supports of μ_S and μ_T, and since the support of the convolution of two measures is the product of their supports, $ST = TS$, and the proof is complete.

PROPOSITION 4.2. *Let G be a topological Hausdorff group of measure preserving transformations acting on a countably additive probability space $(\Omega, \mathfrak{F}, P)$ and suppose that the action is jointly measurable. Suppose, too, that for each $i \in N^d$, $H(i)$ is a compact subgroup of G and that $H(i \vee j)$ is the permutable product of $H(i)$ and $H(j)$ for all $i, j \in N$. Then for each $X \in \mathfrak{L} \log^{d-1} \mathfrak{L}$, $\int X(g(\omega))d_i(g)$ converges for P-almost all ω to $X_\infty(\omega)$ as $i \to \infty$, where $d_i(g)$ refers to normalized Haar measure on $H(i)$, and where X_∞ is the conditional expectation of X with respect to the σ-field of $\mathfrak{F}$-events which are invariant under $\bigcup_i H(i)$. Moreover, if for each $1 \leqslant r \leqslant d$ the σ-field $\mathfrak{F}_{r,\infty}$ is P-trivial, where $\mathfrak{F}_{r,\infty}$ consists of $\mathfrak{F}$-events invariant under all subgroups $H(i)$ for i belonging to the rth axis then for P-almost all ω, the convergence is rearrangeable and $X_\infty(\omega) = E(X)$.*

PROOF. Apply Proposition 4.1, the convergence theorem of Gut [1976] mentioned in §2, and Theorem 4.2.

Let $Y = \{Y_{n,m}, n, m \in N\}$ be a two-parameter, real-valued stochastic process. Associated with Y is the one-parameter, vector-valued, stochastic process $Y^{(1)}$, where, for each n, $Y_n^{(1)}$ is the row sequence $Y_{n,\cdot}$. Also associated with Y is the one-parameter, vector-valued, stochastic process $Y^{(2)}$ where, for each m, $Y_m^{(2)}$ is the column sequence $Y_{\cdot,m}$. Call Y *semiexchangeable* if $Y^{(1)}$ is an exchangeable, vector-valued, stochastic process, and Y_2 is, too. The notion that Y is semiexchangeable can be introduced in an equivalent manner, thus. Let $H(n, 1)$ be the subgroup of permutations of N^2 consisting of the $n!$ permutations which act by permuting the first n columns and by leaving all other columns fixed. Let $H(1, m)$ be defined similarly in terms of rows. For $i = (n, m)$, let $H(i)$ be the product of $H(n, 1)$ and $H(1, m)$, which is a permutable product, and let G be the union of $H(i)$ over $i \in N^2$. Let $\Omega = R^{N^2}$, $\mathfrak{F}$ the usual σ-field on Ω, that is the one generated by the evaluation mappings. Then G acts in an obvious way on Ω, and a process Y is semiexchangeable if, and only if, the law of Y on $\mathfrak{F}$ is G-invariant. Of course, the notion of a stochastic process being semiexchangeable is easily extended to d-parameter, real-valued, stochastic processes Y. For such a process $Y = (Y_i, i \in N^d)$, define

$$X_i = \sum_{j \leqslant i} Y_j / |i|$$

where $|i| = \Pi_r\, i(r)$ is the number of j with $j \leqslant i$.

COROLLARY 4.1. *If* $Y = (Y_i,\ i \in N^d)$ *is semiexchangeable and in* $\mathfrak{L} \log^{d-1} \mathfrak{L}$, *then* X_i *converges almost surely.*

As a special case of a semiexchangeable process, let the Y_i be i.i.d. The zero-one law of Hewitt-Savage [1955] then implies that each $\mathfrak{F}_{r,\infty}$ is P-trivial, so Proposition 4.2 applies to yield the strong law of large numbers for multiparameter arrays due to Zygmund [1951], Smythe [1973]: If $E(|Y_i|\log^+|Y_i|)^{d-1} < \infty$, then $X_i - EX_i$ converges rearrangeably almost surely to zero. The convergence theorem for generalized U-statistics of Sen [1977] could also be seen as a corollary to Proposition 4.2.

REFERENCES

1. R. C. Baker (1976), *Riemann sums and Lebesgue integrals*, Quart. J. Math. Oxford Ser. (2) **27**, 191–198.

2. R. Cairoli (1970), *Une inégalité pour martingales à indices multiples et ses applications*, Seminaire de Probabilités IV, Lecture Notes in Math., vol. 124, Springer-Verlag, Berlin, pp. 1–28.

3, R. Cairoli and J. B. Walsh (1975), *Stochastic integrals in the plane*, Acta Math. **134**, 111–183.

4. C. Dellacherie and P. A. Meyer (1975), *Probabilités et potentiel*, Hermann, Paris.

5. J. Dieudonné (1950), *Sur un théorème de Jessen*, Fund. Math. **37**, 242–248.

6. Lester E. Dubins (1977), *Measurable, tail disintegrations of the Haar integral are purely finitely additive*, Proc. Amer. Math. Soc. **62**, 34–36.

7. N. Dunford (1951), *An individual ergodic theorem for non-commutative transformations*, Acta Sci. Math. (Szeged) **14**, 1–4.

8. R. F. Gundy and N. Th. Varopoulos (1975), *A martingale that occurs in harmonic analysis*, Note No. 157, Analyse Harmonique d'Orsay, Université Paris XI.

9. A. Gut (1976), *Convergence of reversed martingales with multidimensional indices*, Duke Math. J. **43**, 269–275.

10. E. Hewitt and L. J. Savage (1955), *Symmetric measures on Cartesian products*, Trans. Amer. Math. Soc. **80**, 470–501.

11. B. Jessen (1934), *On the approximation of Lebesgue integrals by Riemann sums*, Ann. of Math. (2) **35**, 248–251.

12. J. von Neumann (1934), *Zum Haarschen mass in Topologischon Grupp.*, Comput. Math. **1**.

13. Jacques Neveu (1975), *Discrete-parameter martingales*, North-Holland, Amsterdam; American Elsevier, New York.

14. L. Pontriagin (1946), *Topological groups*, Princeton Univ. Press, Princeton, N.J.

15. Walter Rudin (1964), *An arithmetic property of Riemann sums*, Proc. Amer. Math. Soc. **15**, 321–324.

16. P. K. Sen (1977), *Almost sure convergence of generalized U-statistics*, Ann. Probability **5**, 287–290.

17. R. T. Smythe (1973), *Strong laws of large numbers for r-dimensional arrays of random variables*, Ann. Probability **1**, 164–170.

18. ______ (1974), *Sums of independent random variables on partially ordered sets*, Ann. Probability 2, **5**, 906–917.

19. A. Zygmund (1951), *An individual ergodic theorem for non-commutative transformations*, Acta Sci. Math. (Szeged) **14**, 103–110.

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